Killara HIGH SCHOOL
MATHEMATICS DEPARTMENT

Centre Number

Student Number

2020 | Mathematics Extension 2

Trial HSC Examination

Date: Monday 10th August, 2020

General Instructions

- Reading time – 10 minutes
- Working time – 3 hours
- Write using blue or black pen
- NESA approved calculators may be used
- Show relevant mathematical reasoning and/or calculations

Total Marks: 100

Section I – 10 marks

- Allow about 15 minutes for this section

Section II – 90 marks

- Allow about 2 hours and 45 minutes for this section

Section I (10 marks)	Multiple Choice	/10
Section II (90 marks)	Question 11	/15
	Question 12	/15
	Question 13	/13
	Question 14	/17
	Question 15	/15
	Question 16	/15
	Total	**/100**

This question paper must not be removed from the examination room.
This assessment task constitutes 30% of the course.

Section I

10 marks
Allow about 15 minutes for this section

Use the multiple-choice sheet for Question 1–10

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1. If $z = \frac{3+4i}{1+2i}$, the imaginary part of z is:

A. -2

B. $-\frac{2}{5}$

C. $-2i$

D. $-\frac{2}{5}i$

2. If the vectors $\underset{\sim}{a} = m\underset{\sim}{i} + 4\underset{\sim}{j} + 3\underset{\sim}{k}$ and $\underset{\sim}{b} = m\underset{\sim}{i} + m\underset{\sim}{j} - 4\underset{\sim}{k}$ are perpendicular, then which of the following values of m are correct?

A. $m = -1$ or $m = 1$

B. $m = -2$ or $m = 0$

C. $m = -2$ or $m = 6$

D. $m = -6$ or $m = 2$

3. If z is any complex number satisfying $|z - 1| = 1$, then which of the following is correct?

A. $\arg(z - 1) = 2\arg(z)$

B. $\arg(z - 1) = \arg(z + 1)$

C. $\arg(z) = 2\arg(z + 1)$

D. $2\arg(z) = \frac{2}{3}\arg(z^2 - z)$

Section I continues on the next page

4. A particle moves along a curve so that its position at time t is given by $\underset{\sim}{x} = \begin{pmatrix} t \\ \frac{1}{2}t^2 \\ \frac{1}{3}t^3 \end{pmatrix}$.

The acceleration at $t = 1$ is:

A. $\underset{\sim}{j} + \underset{\sim}{k}$

B. $\underset{\sim}{j} + 2\underset{\sim}{k}$

C. $2\underset{\sim}{j} + \underset{\sim}{k}$

D. $\underset{\sim}{i} + \underset{\sim}{j} + 2\underset{\sim}{k}$

5. A man walks a distance of 3 units from the origin in the direction of $N45°E$, and then walks a distance of 4 units in the direction of $N45°W$ arriving at the point B. The position of B in the Argand plane is:

A. $3e^{\frac{i\pi}{4}} + 4i$

B. $(3 - 4i)e^{\frac{i\pi}{4}}$

C. $(3 + 4i)e^{\frac{i\pi}{4}}$

D. $(4 + 3i)e^{\frac{i\pi}{4}}$

6. Using the substitution $x = \pi - y$, the definite integral $\int_0^{\pi} x \sin x \; dx$ will simplify to:

A. 0

B. $\dfrac{\pi^2}{4}$

C. $\dfrac{\pi}{2}\displaystyle\int_0^{\pi} \sin x \; dx$

D. $\displaystyle\int_0^{\pi} \sin x \; dx$

Section I continues on the next page

7. Which of the following statements is a negation of the following statement?

$$\forall x \in R^+, \exists y \in R^+ : xy = 1$$

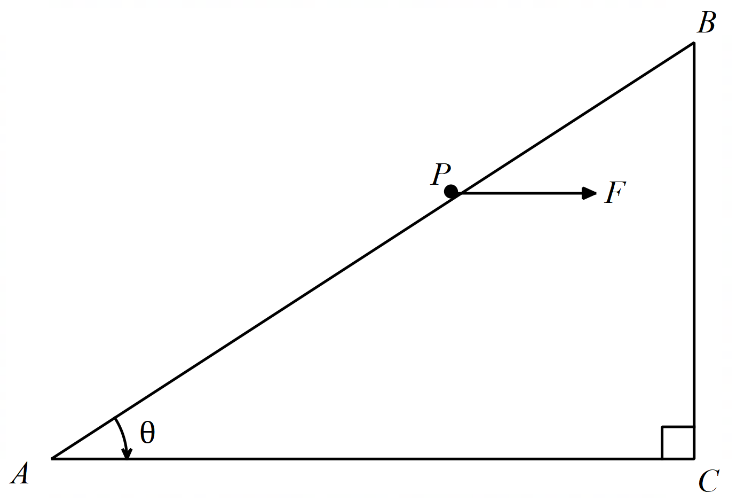
A. $\exists x, y \in R^+$ such that $xy \neq 1$

B. $\forall x, y \in R^+, \quad xy \neq 1$

C. $\exists x \in R^+ : \forall y \in R^+, xy \neq 1$

D. $\forall x \in R^+$, there exists a positive real number y such that $xy \neq 1$

8. A horizontal force F Newtons is applied to a small object P of mass m kg on a smooth plane, inclined to the horizontal at an angle θ.



If F is just enough to keep P in equilibrium, then the magnitude of F is:

A. $mg\cos^2\theta$

B. $mg\sin^2\theta$

C. $mg\cos\theta$

D. $mg\tan\theta$

Section I continues on the next page

9. Which of the following is **false**?

A. $\displaystyle\int_{-3}^{3} x^3 e^{-x^2} dx = 0$

B. $\displaystyle\int_{-4}^{4} \frac{x^2}{x^2+4} dx = 2\int_{0}^{4} \frac{x^2}{x^2+4} dx$

C. $\displaystyle\int_{0}^{\pi} \sin^4\theta \, d\theta > \int_{0}^{\pi} \sin 4\theta \, d\theta$

D. $\displaystyle\int_{0}^{1} x^4 dx < \int_{0}^{1} x^5 dx$

10. $\underset{\sim}{a}, \underset{\sim}{b}$ and $\underset{\sim}{c}$ are vectors of magnitude 3, 4 and 5 respectively.

It is also given that:

$\underset{\sim}{a}$ is perpendicular to $\left(\underset{\sim}{b} + \underset{\sim}{c}\right)$,

$\underset{\sim}{b}$ is perpendicular to $\left(\underset{\sim}{c} + \underset{\sim}{a}\right)$ and

$\underset{\sim}{c}$ is perpendicular to $\left(\underset{\sim}{a} + \underset{\sim}{b}\right)$.

Then, the magnitude of the vector $\underset{\sim}{a} + \underset{\sim}{b} + \underset{\sim}{c}$ is:

A. 5

B. $5\sqrt{2}$

C. $5\sqrt{3}$

D. 12

End of Section I

Section II

Answer each question in the appropriate writing booklet. Extra writing booklets are available.

In Questions 11 – 16, your response should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use the Question 11 Writing Booklet.

(a) Let $\omega_1 = 8 - 2i$ and $\omega_2 = -5 + 3i$. **2**
Find:

$$\omega_1 + \overline{\omega_2}$$

(b) (i) Express $z = \sqrt{2} - i\sqrt{2}$ in the exponential form **2**

(ii) Hence, write z^{22} in the $a + ib$ form where $a, b \in R$ **2**

(c) (i) Find the square roots of $-35 + 12i$ **3**

(ii) Solve $z^2 - (5 + 4i)z + 11 + 7i = 0$ **2**

(d) Find **2**

$$\int \frac{dx}{x(\ln x)^2}$$

(e) Find **2**

$$\int \frac{1}{x^2 - 6x + 13} dx$$

End of Question 11

Question 12 (15 marks) Use the Question 12 Writing Booklet.

(a) Prove the following statement using a proof by contradiction. **4**

"For each irrational number s, the number $2s + 1$ is also irrational"

(b) Show that **4**

$$\int_1^3 \frac{6t + 23}{(2t - 1)(t + 6)} dt = \ln \frac{225}{7}$$

(c) Evaluate **3**

$$\int_0^{\frac{\pi}{2}} \frac{d\theta}{2 + cos\theta}$$

by using the substitution $t = tan\frac{\theta}{2}$

(d) (i) Show that if $1, \omega_1, \omega_2$ are the cube roots of 1, **1**

$$1 + \omega_1 + {\omega_1}^2 = 0$$

and

$$1 + \omega_2 + {\omega_2}^2 = 0$$

(ii) If n is not a multiple of 3, prove that **3**

$x^{2n} + x^n + 1$ is divisible by $x^2 + x + 1$

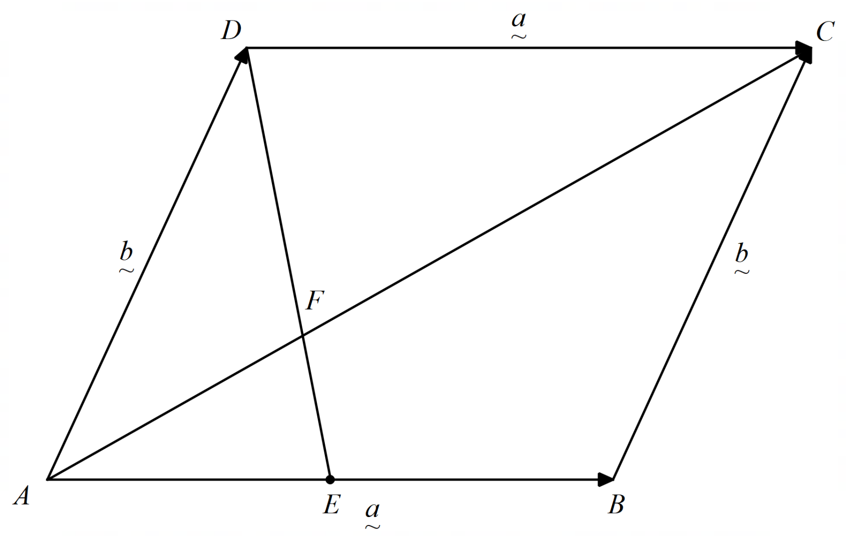
End of Question 12

Question 13 (13 marks) Use the Question 13 Writing Booklet.

(a) Prove by mathematical induction that $\forall n \in Z^+$ **3**

$$\sum_{i=1}^{n} \sqrt{i} > \frac{2n\sqrt{n}}{3}$$

(b) $ABCD$ is a parallelogram and E is the midpoint of AB.



Using vectors, show that any line joining any vertex of a parallelogram to the midpoint of a side not passing through that vertex divides the opposite diagonal in the ratio 1: 2. (i.e. Show DE divides AC, in the ratio 1: 2) **4**

(c) The acceleration of a particle moving along the x-axis is given by

$$\frac{d^2x}{dt^2} = 2x^3 - 10x$$

(i) If the particle starts at the origin with velocity u, show that its velocity is given by $v^2 - u^2 = x^4 - 10x^2$ **2**

(ii) If $u = 3$, show that the particle oscillates within the interval $-1 \leq x \leq 1$ **3**

(iii) Is the motion referred to in (ii), an example of simple harmonic motion? Give clear reason for your answer. **1**

End of Question 13

Question 14 (17 marks) Use the Question 14 Writing Booklet.

(a) (i) Show that $a^2 + b^2 > 2ab$, where a and b are distinct positive real numbers **1**

(ii) Hence, show that $a^2 + b^2 + c^2 > ab + bc + ca$, where a, b and c are distinct positive real numbers **2**

(iii) Hence, or otherwise prove that **2**

$$\frac{a^2b^2 + b^2c^2 + c^2a^2}{a + b + c} > abc,$$

Where a, b and c are distinct positive real numbers

(b) (i) If $Z = \cos\theta + i\ \sin\theta$, **1**
Express $Z^n + Z^{-n}$ in terms of θ

Let, for real values of k,

$$f(\theta) = 1 + k\ \cos\theta + k^2\cos 2\theta + \ k^3\cos 3\theta + \cdots + \ k^n\cos n\theta + \cdots$$

(ii) Using the result in (i) and expressing $f(\theta)$ as the sum of two geometric progressions, prove that **3**

$$f(\theta) = \frac{1 - k\cos\theta}{1 - 2k\cos\theta + k^2}, \qquad |k| < 1$$

(iii) Verify the result in (ii) for $|k| < 1$ and $\theta = \frac{\pi}{2}$ **1**

Question 14 continues on the next page

Question 14 (continued)

(c) $\underset{\sim}{r_1}$ and $\underset{\sim}{r_2}$ are two lines with vector equations:

$$\underset{\sim}{r_1} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \text{ and}$$

$$\underset{\sim}{r_2} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}, \qquad \lambda, \mu \in R$$

(i) Show that these two lines intersect. **3**

(ii) Find the angle between the lines. **1**

(iii) Find the shortest distance from the point $P(1,2,0)$ to the line **3**

$$\underset{\sim}{r_1} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

End of Question 14

Question 15 (15 marks) Use the Question 15 Writing Booklet.

(a) If the functions $f(x)$ and $g(x)$ are such that $f(x) > g(x) \geq 0$ for $a \leq x \leq b$, by using a sketch (or otherwise) explain why **2**

$$\int_a^b f(x)\,dx > \int_a^b g(x)\,dx$$

(b) Let

$$I_n = \int_0^1 (1-t^2)^{\frac{n-1}{2}}\,dt,$$

Where, n is a non-negative integer.

(i) Using integration by parts, or otherwise, show that **4**

$$n\,I_n = (n-1)\,I_{n-2} \quad \text{if } n \geq 2$$

(ii) Let $J_n = n\,I_n\,I_{n-1}, \quad n \geq 1.$ **4**

Show that

$$J_n = \frac{\pi}{2}, \quad \forall n \geq 1$$

(iii) Using part (a), or otherwise, show that **3**

$$0 < I_n < I_{n-1}$$

(iv) Hence, or otherwise, prove that **2**

$$\sqrt{\frac{\pi}{2n+2}} < I_n < \sqrt{\frac{\pi}{2n}}$$

End of Question 15

Question 16 (15 marks) Use the Question 16 Writing Booklet.

(a) When a jet aircraft touches down, two different retarding forces combine to bring it to rest. If the jet has a mass of M kg and a speed of v m/s, there is a constant frictional force of $\frac{1}{4}M$ newtons and a force of $\frac{1}{108}Mv^2$ newtons due to the reverse thrust of the engines.
The reverse thrust of the engines do not take into effect until 20 seconds after touch down.

(i) Show that **2**

$$\frac{d^2x}{dt^2} = -\frac{1}{4} \quad \text{for } 0 < t \leq 20$$

And that for $t > 20$, and until after the jet stops,

$$\frac{d^2x}{dt^2} = -\frac{1}{108}(27 + v^2)$$

(ii) If the jet's speed at touch down is 60 m/s, show that $v = 55$ and $x = 1150$ at the instant the reverse thrust of the engines takes effect. **2**

(iii) Show that when $t > 20$, **2**

$$x = 1150 + 54\,\{\ln(27 + 55^2) - \ln(27 + v^2)\}$$

(iv) Calculate how far from the touchdown point the jet comes to rest. Give your answer to the nearest metre. **1**

Question 16 continues on the next page

(b) By considering the sum to n terms of the series

$$1 - t + t^2 - t^3 + \cdots + (-1)^{n-1}t^{n-1}, \qquad t \neq -1$$

(i) Show that **2**

$$x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots + (-1)^n \frac{x^n}{n} = \log_e(1 + x) - (-1)^n \int_0^x \frac{t^n}{1+t}\,dt$$

for $0 < t < x$,

(ii) Also, show that **2**

$$\int_0^x \frac{t^n}{1+t}\,dt < \int_0^x t^n\,dt$$

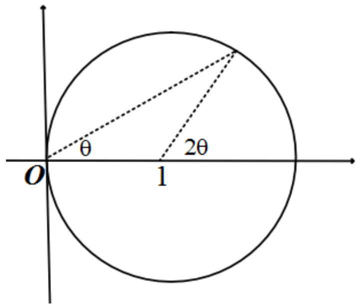
(iii) For $0 < x \leq 1$, show that **2**

$$\int_0^x \frac{t^n}{1+t}\,dt < \frac{1}{n+1}$$

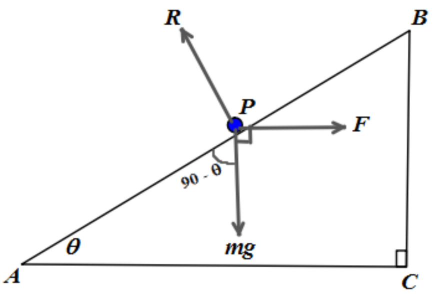
(iv) Hence, prove that as $n \to \infty$, **2**

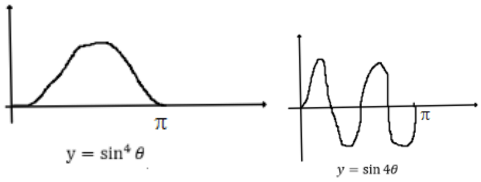
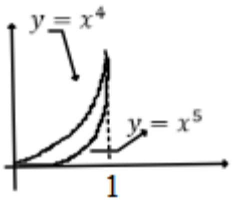
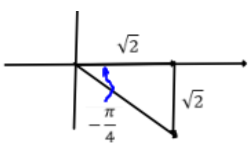
$$x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots + (-1)^n \frac{x^n}{n} + \cdots = \log_e(1 + x)$$

End of Examination

<table>
<tr><td colspan="2">Multiple Choice</td><td></td><td></td></tr>
<tr><td>1.</td><td>$z = \frac{3+4i}{1+2i}$
$= \frac{(3+4i)(1-2i)}{(1+2i)(1-2i)}$
$= \frac{11}{5} + \left(-\frac{2}{5}\right) i$
$Im(z) = -\frac{2}{5}$</td><td>B</td><td></td></tr>
<tr><td>2.</td><td>$\underset{\sim}{\boldsymbol{a}} = m\underset{\sim}{\boldsymbol{i}} + 4\underset{\sim}{\boldsymbol{j}} + 3\underset{\sim}{\boldsymbol{k}}$ and
$\underset{\sim}{\boldsymbol{b}} = m\underset{\sim}{\boldsymbol{i}} + m\underset{\sim}{\boldsymbol{j}} - 4\underset{\sim}{\boldsymbol{k}}$
If $\underset{\sim}{\boldsymbol{a}}$ and $\underset{\sim}{\boldsymbol{b}}$ are perpendicular, $\underset{\sim}{\boldsymbol{a}}.\underset{\sim}{\boldsymbol{b}}$ = 0
$$m^2 + 4m - 12 = 0$$
$$(m+6)(m-2) = 0$$
$$m = -6, 2$$</td><td>D</td><td></td></tr>
<tr><td>3.</td><td>

Angle subtended at the centre is double the angle at the circumference</td><td>A</td><td></td></tr>
<tr><td>4.</td><td>$\underset{\sim}{\boldsymbol{r}} = \begin{pmatrix} \boldsymbol{t} \\ \frac{1}{2}\boldsymbol{t}^2 \\ \frac{1}{3}\boldsymbol{t}^3 \end{pmatrix}$; $\dot{\underset{\sim}{\boldsymbol{r}}} = \begin{pmatrix} \boldsymbol{1} \\ \boldsymbol{t} \\ \boldsymbol{t}^2 \end{pmatrix}$; $\ddot{\underset{\sim}{\boldsymbol{r}}} = \begin{pmatrix} \boldsymbol{0} \\ \boldsymbol{1} \\ \boldsymbol{2t} \end{pmatrix}$
At t = 1, $\ddot{\underset{\sim}{\boldsymbol{r}}} = \begin{pmatrix} \boldsymbol{0} \\ \boldsymbol{1} \\ \boldsymbol{2} \end{pmatrix}$ = $\underset{\sim}{\boldsymbol{j}} + 2\underset{\sim}{\boldsymbol{k}}$</td><td>B</td><td></td></tr>
<tr><td>5.</td><td>Suppose the man reaches A after walking 3 units in $N45°E$ and B after walking a distance of 4 units in $N45°W$. Position of A in the Argand diagram is $3e^{\frac{i\pi}{4}}$. Let the position of B be z. Since $\angle OAB = \frac{\pi}{2}$, we have
$$\arg\left(\frac{0 - 3e^{\frac{i\pi}{4}}}{z - 3e^{\frac{i\pi}{4}}}\right) = \frac{\pi}{2}$$
$$\frac{0 - 3e^{\frac{i\pi}{4}}}{z - 3e^{\frac{i\pi}{4}}} = \frac{OA}{AB}(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2})$$
$$\frac{0 - 3e^{\frac{i\pi}{4}}}{z - 3e^{\frac{i\pi}{4}}} = \frac{3i}{4}$$
$$z - 3e^{\frac{i\pi}{4}} = 4ie^{\frac{i\pi}{4}}$$
$$z = (3+4i)\, e^{\frac{i\pi}{4}}$$</td><td>C</td><td></td></tr>
</table>

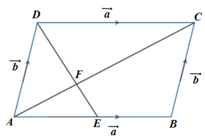
6.	$$\int_0^{\pi} x\sin x\ dx = \int_0^{\pi} x\sin x\ dx$$ $x = \pi - y$ $dx = -dy$ $x = 0, y = \pi$ $x =, y = 0$ $$\int_0^{\pi} x\sin x\ dx = \int_{\pi}^{0} (\pi - y)\sin(\pi - y) \times -dy$$ $$= \int_0^{\pi} (\pi - y)\sin y\, dy$$ $$= \int_0^{\pi} \pi \sin y\, dy - \int_0^{\pi} y\sin y\, dy$$ $$\int_0^{\pi} x\sin x\ dx = \int_0^{\pi} y\sin y\, dy$$ $$2\int_0^{\pi} y\sin y\, dy = \pi\int_0^{\pi} \sin y\, dy$$ $$\int_0^{\pi} y\sin y\, dy = \frac{\pi}{2}\int_0^{\pi} \sin y\, dy$$	C	
7.	C $\exists x \in R^+ : \forall y \in R^+, xy \neq 1$	C	
8.	 By Lami's Theorem, $$\frac{R}{\sin 90°} = \frac{F}{\sin(180° - \theta)} = \frac{mg}{\sin(90° + \theta)}$$ $$\frac{R}{1} = \frac{F}{sin\theta} = \frac{mg}{cos\theta}$$ $$F = mg\ tan\theta$$	D	

9.	$(A)\ \int_{-3}^{3} x^3 e^{-x^2} dx = 0 \quad odd\ function, \qquad True$ (B) Even function, True (C)  True (D)  False	D	
10.	$\left\|\vec{a}+\vec{b}+\vec{c}\right\|^2 = \left(\vec{a}+\vec{b}+\vec{c}\right).\left(\vec{a}+\vec{b}+\vec{c}\right)$ $= \vec{a}.\vec{a} + \vec{a}.\left(\vec{b}+\vec{c}\right) + \vec{b}.\vec{b} + \vec{b}.\left(\vec{a}+\vec{c}\right) +$ $\vec{c}.\vec{c} + \vec{c}.\left(\vec{a}+\vec{b}\right)$ $= \left\|\vec{a}\right\|^2 + \vec{a}.\left(\vec{b}+\vec{c}\right) + \left\|\vec{b}\right\|^2 + \vec{b}.\left(\vec{a}+\vec{c}\right) +$ $\left\|\vec{c}\right\|^2 + \vec{c}.\left(\vec{a}+\vec{b}\right)$ $= 9 + 0 + 16 + 0 + 25 + 0$ $= 50$ $\left\|\vec{a}+\vec{b}+\vec{c}\right\| = \sqrt{50} = 5\sqrt{2}$	B	
Question 11			
(a)	$\omega_1 = 8 - 2i$ and $\omega_2 = -5 + 3i$ $\omega_1 = 8 - 2i$ and $w_1 + \overline{w_2} = 8 - 2i - 5 - 3i$ $3 - 5i$	1 mark: correct $\overline{w_2}$ 1 mark: correctly adds ω_1 and $\overline{w_2}$	
(b)(i)	 $z = \sqrt{2} - i\sqrt{2}$ $\|z\| = \sqrt{2+2} = 2$ $\arg(z) = -\frac{\pi}{4}$ $z = 2\, e^{\frac{-i\pi}{4}}$	1 mark: correct $\|z\|$ or correct $\arg(z)$ 1 mark: correct exponential form	Well done

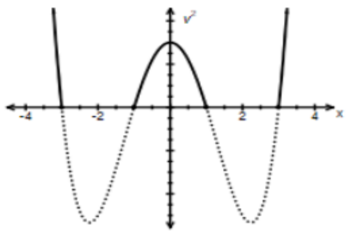
(ii)	$z^{22} = 2^{22}\, e^{\frac{-i22\pi}{4}}$ $z^{22} = 2^{22}\, e^{\frac{-i11\pi}{2}}$ $z^{22} = 2^{22}\left(\cos\left(-\frac{11\pi}{2}\right) + isin\left(-\frac{11\pi}{2}\right)\right)$ $z^{22} = 2^{22}\left(\cos\left(\frac{11\pi}{2}\right) - isin\left(\frac{11\pi}{2}\right)\right)$ $z^{22} = 2^{22}\big(0 - (-i)\big)$ $z^{22} = 2^{22}i$	1 mark: correctly writes the expression for z^{22} 1 mark: correctly evaluates and gives the answer in the simplest form	Some minor in calculation of $sin\left(\frac{11\pi}{2}\right)$
(c)(i)	Let $\sqrt{-35+12i} = a + ib$ $a^2 - b^2 = -35$ $a^2 + b^2 = \sqrt{(-35)^2 + 12^2}$ = 37 $a^2 = 1 \quad \therefore a = \pm 1$ $b^2 = 36 \quad \therefore b = \pm 6$ $\sqrt{-35+12i} = \pm(1 + 6i)$	1 mark: gives the correct values for $a^2 - b^2$ and $2ab = 12$ 2 mark: correctly evaluates a and b and gives the square root. 1 mark: Minor error	Well done (multiple methods used)
(c)(ii)	$z^2 - (5+4i)z + 11 + 7i = 0$ $z = \dfrac{(5+4i) \pm \sqrt{(5+4i)^2 - 4(11+7i)}}{2}$ $z = \dfrac{(5+4i) \pm \sqrt{-35+12i}}{2}$ Using (c)(i), $z = \dfrac{(5+4i) \pm (1+6i)}{2}$ $z = \dfrac{5+4i+1+6i}{2}, \dfrac{5+4i-1-6i}{2}.$ $\frac{5+4i+1-6i}{2}, \quad \frac{5+4i-1+6i}{2}$ $z = 3 - i, \ \ 2 + 5i$	1 mark: applies quadratic formula correctly 1 mark: Uses the solution from (c)(i) to give all the correct roots	Well done (other than careless errors)
(d)	$I = \int \frac{dx}{x(\ln x)^2}$ Let $u = \ln x$ $\frac{du}{dx} = \frac{1}{x}$ $du = \frac{1}{x}dx$ $I = \int \frac{du}{u^2}$ $= -\frac{1}{u} + c$ $= -\frac{1}{\ln x} + c$	2 mark: Correct answer from correct working 1 mark: Minor error	Well done
(e)	$\int \frac{1}{x^2 - 6x + 13}dx$ $\int \frac{1}{(x-3)^2 + 4}dx$ $= \frac{1}{2}\tan^{-1}\frac{x-3}{2} + C$	2 mark: Correct answer from correct working 1 mark: Minor error	Well done

Question 12			
(a)	The statement to prove is: $\forall s \in R, if\ s \notin Q, then\ 2s+1 \notin Q$ Proof: Suppose the statement is false. That is $\exists s \in R : s \notin Q$ and $2s+1 \in Q$. 1 mark Then, $2s+1 = \frac{a}{b}$ for some $a, b \in J$ and $b \neq 0$. $2s = \frac{a}{b} - 1 = \frac{a-b}{b}$ $s = \frac{a-b}{2b}$ $\therefore s = \frac{c}{d}$ for some $c, d \in J$ and $d \neq 0$. *However,* $s \notin Q$. Thus we have reached a contradiction in our assumption that the original statement was false. Thus the statement is true.	1 mark: Gives proof with correct notation and fluid logic- must demonstrate mastery of mathematical notation 1 mark: Writes the statement of contradiction 2 marks: for correct proof (demonstrates the contradiction and states the meaning of each statement. 1 mark: Gives the correct proof with not-so-good explanation	Poorly done. Try to uses as much mathematical notation as possible. (rewrite the given statement using mathematical notation) Please see the contradiction statement: most students $\forall s$ instead of $\exists s \in R$
(b)	$\int_1^3 \frac{6t+23}{(2t-1)(t+6)}dt$ $\frac{6t+23}{(2t-1)(t+6)} = \frac{A}{2t-1} + \frac{B}{t+6}$ $A = 4, B = 1$ $\int_1^3 \frac{6t+23}{(2t-1)(t+6)}dt = \int_1^3 \frac{4}{2t-1} + \frac{1}{t+6}dt$ $= \int_1^3 \frac{2\times 2}{2t-1} + \frac{1}{t+6}dt$ $[\ln((2t-1)^2(t+6)]_1^3$ $= \ln\frac{25\times 9}{1\times 7}$ $= \ln\frac{225}{7}$	1 mark: correctly converts to partial fractions 2 mark: correct integration (1 mark: Minor error in integration) 1 mark: correct evaluation	Well done
(c)	$\int_0^{\frac{\pi}{2}} \frac{d\theta}{2+cos\theta}$ $t = \tan\frac{\theta}{2}$ $dt = \frac{1}{2}\sec^2\frac{\theta}{2}dt = (1+t)^2dt$ $\theta = 0, \quad t = 0$ $\theta = \frac{\pi}{2}, \quad t = 1$ **1 mark** $\int_0^1 \frac{\frac{2}{1+t^2}dt}{2+\frac{1-t^2}{1+t^2}}$ $\int_0^1 \frac{2dt}{2(1+t^2)+1-t^2}$ $\int_0^1 \frac{2dt}{3+t^2}$ **1 mark** $\left[\frac{2}{\sqrt{3}}\tan^{-1}\frac{t}{\sqrt{3}}\right]_0^1$ 2 mark : correct integration and evaluation $\frac{2}{\sqrt{3}} \times \left(\frac{\pi}{6} - 0\right)$ $\frac{\pi}{3\sqrt{3}}$ 1 mark:	1 mark: correctly converts the limits and $d\theta$ in terms t 1 mark: correctly converts the integrand 2 marks: correctly integrates and evaluates 1 mark: minor error in integration and evaluation	Well done, except for error in simplification of the integrand and making the converted from far too simple

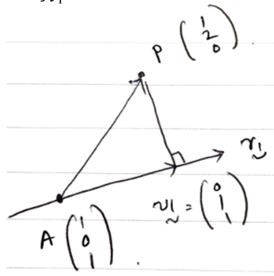
(d)(i)	Since, ω_1 is a cube root of unity, $\omega_1^3 - 1 = 0$. Ie. $(\omega_1 - 1)(\omega_1^2 + \omega_1 + 1) = 0$. As $\omega_1 \neq 0$, $$(\omega_1^2 + \omega_1 + 1) = 0$$ Also, $$(\omega_2^2 + \omega_2 + 1) = 0$$	1 mark: factorises $\omega_1^3 - 1 = 0$ and explains the results with correct reasons.	Well done
(ii)	Let $\alpha = \omega_1^n$. Then $\alpha \neq 1$ and $\alpha^3 = (\omega_1^n)^3 = (\omega_1^3)^n = 1$ $\therefore$ α is a cube root of 1. **1 mark** By (a), $\alpha^2 + \alpha + 1 = 0$ Thus $\omega_1^{2n} + \omega_1^n + 1 = 0$ Ie. ω_1 Is a root of $x^{2n} + x^n + 1$ Similarly if $\beta = \omega_2^n$, $\beta^2 + \beta + 1 = 0$, $$\omega_2^{2n} + \omega_2^n + 1 = 0$$ ω_2 Is a root of $x^{2n} + x^n + 1$ From (i), using remainder theorem, 1 $$x^2 + x + 1 = (x - \omega_1)(x - \omega_2)$$ But we have $\omega_1^{2n} + \omega_1^n + 1 = 0$ and $\omega_2^{2n} + \omega_2^n + 1 = 0$ 2 ω_1 and ω_2 are roots of $x^{2n} + x^n + 1 = 0$ $\therefore$ $(x - \omega_1)$ and $(x - \omega_2)$ are factors of 3 $x^{2n} + x^n + 1$ Thus, $x^{2n} + x^n + 1$ is divisible by $x^2 + x + 1$ 4	1 mark: proves ω^n is a root of $x^3 - 1 = 0$ 1 mark: explains α and β are roots	Poorly done. Many students used the method of representing $n = 3k + 1$ or $n = 3k + 2$ which is fine. However, when you substitute ω_1 into $x^{2(3k+1)} + x^{(3k+1)} + 1$ and prove that it satisfies the equation as $(\omega_1^2 + \omega_1 + 1) = 0$, you are only proving ω_1 is a root. Statements 1, 2, 3 and 4 must be given to complete the proof The question is about proving ω_1 and ω_2 are roots of $x^{2n} + x^n + 1 = 0$
Question 13			
(a)	$$\sum_{i=1}^{n} \sqrt{i} > \frac{2n\sqrt{n}}{3}$$ Step 1: For n = 1, $$\sqrt{1} > \frac{2 \times \sqrt{1}}{3}$$ $1 > \frac{2}{3}$ True for n = 1 ☑ Inductive step:	1 mark: proves basic step 2 mark: completely correct proof for the inductive step and gives a conclusion	Most students received 2 out 3 for this question. Inductive step was not shown clearly with correct logic

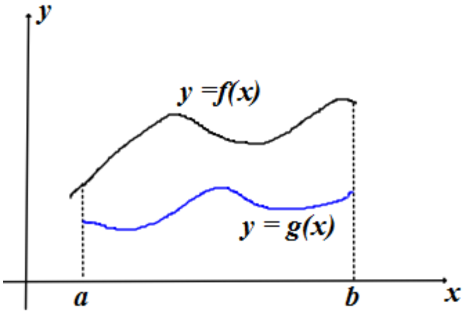
	Suppose the result is true for n = k, then we prove it is true for n = k+1 Thus, $\sum_{i=1}^{k} \sqrt{i} > \frac{2k\sqrt{k}}{3}$, prove $$\sum_{i=1}^{k+1} \sqrt{i} > \frac{2(k+1)\sqrt{k+1}}{3}$$ $$\sum_{i=1}^{k+1} \sqrt{i} = \sum_{i=1}^{k} \sqrt{i} + \sqrt{i+1}$$ $$\geq \frac{2k\sqrt{k}}{3} + \sqrt{i+1}$$ We'll prove that $$\frac{2k\sqrt{k}}{3} + \sqrt{k+1} > \frac{2(k+1)\sqrt{k+1}}{3}$$ $$\frac{2k\sqrt{k} + 3\sqrt{k+1}}{3} > \frac{2(k+1)\sqrt{k+1}}{3}$$ $$2k\sqrt{k} + 3\sqrt{k+1} > 2(k+1)\sqrt{k+1}$$ $$2k\sqrt{k} > (2k-1)\sqrt{k+1}$$ $$4k^2k > (2k-1)^2(k+1)$$ $$4k^3 > (4k^2 - 4k + 1)(k+1)$$ $$4k^3 > (4k^3 - 3k + 1)$$ $0 > -3k + 1 \quad \forall k \geq 1 \quad k \in Z^+$ ☑ ☑ Thus the result is true for n = k, then it is true for n = k+1 Hence by principle of mathematical induction, true for all $n \geq 1$	1 mark: progress toward the inductive proof	
(b)	Let ABCD be the parallelogram $\vec{AB} = \vec{DC} = \vec{a}$ $\vec{AD} = \vec{BC} = \vec{b}$ Let $\vec{AF} = x\vec{AC}$ and $\vec{EF} = y\,\vec{ED}$ $\vec{AE} = \frac{\vec{a}}{2}$, $\quad \vec{ED} = \vec{b} - \frac{\vec{a}}{2}$, $\vec{EF} = y\left(\vec{b} - \frac{\vec{a}}{2}\right)$, $\vec{AF} = \frac{\vec{a}}{2} + y\left(\vec{b} - \frac{\vec{a}}{2}\right)$ $= x\left(\vec{a} + \vec{b}\right)$ ☑ $x\left(\vec{a} + \vec{b}\right) = \frac{\vec{a}}{2} + y\left(\vec{b} - \frac{\vec{a}}{2}\right)$ ☑ Equating coefficients of $\vec{a}$ and $\vec{b}$, we get $x = \frac{1}{2}(1-y)$ and $x = y$	 1 mark: Writes $\vec{AF} = x\left(\vec{a} + \vec{b}\right)$ 1 mark: writes $\vec{AF} = \frac{\vec{a}}{2} + y\left(\vec{b} - \frac{\vec{a}}{2}\right)$	

	Thus, $$y = \frac{1}{2}(1 - y)$$ $$y = \frac{1}{3} = x$$ Hence, $\overrightarrow{AF} = \frac{1}{3}\ \overrightarrow{AC};\quad \overrightarrow{EF} = \frac{1}{3}\ \overrightarrow{ED};$ $AF: FC = 1: 2$ $EF: FD = 1: 2$ ☑	1 mark: equates the coefficients of $\vec{a}$ and $\vec{b}$, and evaluates y and hence the result	

(c)(i)	$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 2x^3 - 10x$ $\therefore \frac{1}{2}v^2 = \frac{x^4}{2} - 5x^2 + c$ **1 mark** When $x = 0,\ v = u \rightarrow c = \frac{1}{2}u^2$ $\therefore \frac{1}{2}v^2 = \frac{x^4}{2} - 5x^2 + \frac{1}{2}u^2$ **1 mark** ☑ $v^2 - u^2 = x^4 - 10x^2$	Applies $\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$ and integrates 1 mark: correctly evaluates the constant of integration and proves the result	Well done
(ii)	If $u = 3$ then $v^2 - 9 = x^4 - 10x^2$ $v^2 = x^4 - 10x^2 + 9$ $= (x^2 - 9)(x^2 - 1)$ $= (x-1)(x+1)(x-3)(x+3)$ **1 mark**  Since, $v^2 \geq 0$ for motion to exist then $(x-1)(x+1)(x-3)(x+3) \geq 0$ and since the particle starts at $x = 0$ with $v = 3$ it is moving to the right. **1 mark** At $x = 1. v = 0$ and $\ddot{x} = -8$ and so the particle will move to the left until it reaches $x = -1$ where $v = 0$ and $\ddot{x} = 8$. This means the particle will then move to the right until $v = 0$ again at $x = 1$. **1 mark** Thus it oscillates in the interval $-1 \leq x \leq 1$	1 mark: correctly factorises and solves the inequality 2 marks: uses the signs of $\dot{x}$ and $\ddot{x}$ to describe the direction of motion and proves the particle oscillates in the interval $-1 \leq x \leq 1$ 1 mark: describes the motion at leaast at one point	Poorly done.
(iii)	Not SHM since $\ddot{x} \neq -n^2(x - b)$	1 mark	
Question 14			
a)(i)	$(a-b)^2 > 0 \quad (a \neq b)$ $\therefore \quad a^2 - 2ab + b^2 > 0$ ☑ $\therefore \quad a^2 + b^2 > 2ab$ -------[1]	1 mark: proves the result	Well done
a)(ii)	Similarly $b^2 + c^2 > 2bc$ -------[2] and $c^2 + a^2 > 2ca$ -------[3] ☑ Now [1] + [2] + [3] gives $2(a^2 + b^2 + c^2) > 2(ab + bc + ca)$ ☑ $\therefore \quad a^2 + b^2 + c^2 > ab + bc + ca$	1 mark: writes the staments 1 mark: adds the statements to prove the result	Well done

a)(iii)	Using the result in (ii) Let $a \to ab;\ b \to bc;\ c \to ca.$ $\therefore\ a^2b^2 + b^2c^2 + c^2a^2 > ab.bc + bc.ca + ca.ab$ ☑ $a^2b^2 + b^2c^2 + c^2a^2 > abc(a+b+c)$ ☑ $\therefore\ \frac{a^2b^2+b^2c^2+c^2a^2}{a+b+c} > abc$	2 marks: proves the result 1 mark: minor error in the proof	Well done
(b)(i)	$Z^n + Z^{-n}$ $= (\cos n\theta + i\sin n\theta) + (\cos(-n\theta) + i\sin(-n\theta))$ $= 2\cos n\theta$ **1 mark**	1 mark: proves the result	Well done
(ii)	$2\,f(\theta) = 2 + k(2cos\theta) + k^2(2cos2\theta) + \cdots. +$ $k^n(2cosn\theta)$ $= 2 + k(z + z^{-1}) + k^2(z^2 + z^{-2}) + \cdots$ $+ k^n(z^n + z^{-n}) + \cdots$ $= (1 + kz + k^2z^2 + \cdots \ldots. + k^nz^n + \cdots.) +$ $(1 + \frac{k}{z} + \left(\frac{k}{z}\right)^2 + \cdots \ldots. + \left(\frac{k}{z}\right)^n + \cdots \ldots)$ Thus, $$f(\theta) = \frac{1}{2}\left[\frac{1}{1-kz} + \frac{1}{1-\frac{k}{z}}\right]$$ $f(\theta) = \frac{1}{2}\left[\frac{1}{1-kz} + \frac{z}{z-k}\right]$ **1 mark** $\frac{1}{1-kz} = \frac{1}{1 - kcos\theta - iksin\theta}$ $= \frac{1 - kcos\theta + iksin\theta}{(1-kcos\theta)^2 + k^2\sin^2\theta}$ $= \frac{1 - kcos\theta + iksin\theta}{1 - 2kcos\theta + k^2}$ $\frac{z}{z-k} = \frac{(cos\theta + isin\theta)}{(cos\theta - k) + isin\theta}$ $= \frac{(cos\theta + isin\theta)[(cos\theta - k) - isin\theta]}{[(cos\theta - k) + isin\theta][(cos\theta - k) + isin\theta]}$ $= \frac{1 - kcos\theta - iksin\theta}{1 - 2kcos\theta + k^2}$ $f(\theta) = \frac{1}{2}\left[\frac{1 - kcos\theta + iksin\theta + 1 - kcos\theta - iksin\theta}{1 - 2kcos\theta + k^2}\right]$ $= \frac{1 - kcos\theta}{1 - 2kcos\theta + k^2}$	1 mark: applies the result from (i) and find the sum of the infinite series 2 marks: Rationalises the denominator and finds the result 1 mark: attempts to simplify the expression	Some students experienced difficulty in rationalising the denominator

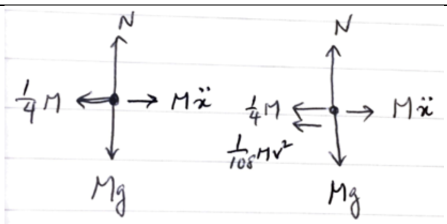
(iii)	For $\lvert k\rvert < 1, \theta = \frac{\pi}{2}$, $1 + k\cos\theta + k^2\cos 2\theta + \cdots + k^n\cos n\theta + \cdots$ $= 1 - k^2 + k^4 - k^6 + \cdots$ $= \frac{1}{1-(-k^2)} = \frac{1}{1+k^2}$ Also, $\frac{1-k\cos\theta}{1-2k\cos\theta+k^2} = \frac{1-k\cos\frac{\pi}{2}}{1-2k\cos\frac{\pi}{2}+k^2} = \frac{1}{1+k^2}$	1 mark: substitutes and verifies the result	Verify means you need to show both results give you the same value when $\theta = \frac{\pi}{2}$
c)(i)	$\underset{\sim}{r_1} = \begin{pmatrix}1\\0\\1\end{pmatrix} + \lambda\begin{pmatrix}0\\1\\1\end{pmatrix} = \begin{pmatrix}1\\\lambda\\1+\lambda\end{pmatrix}$, $\underset{\sim}{r_2} = \begin{pmatrix}2\\0\\0\end{pmatrix} + \mu\begin{pmatrix}1\\3\\2\end{pmatrix} = \begin{pmatrix}2+\mu\\3\mu\\2\mu\end{pmatrix}$ For point of intersection to occur, $\begin{pmatrix}1\\\lambda\\1+\lambda\end{pmatrix} = \begin{pmatrix}2+\mu\\3\mu\\2\mu\end{pmatrix}$ $2+\mu = 1 \rightarrow \mu = -1$ $\lambda = 3\mu = -3$ Substitute in $1+\lambda = 2\mu \rightarrow 1-3 = 2\times -1$ *True.* Hence, the two lines intersect	1 mark: Equates the parametric equations, finds the values of λ and μ. 1 mark: Substitutes into the third equation to prove true and thus proving the intersection.	Well done
c)(ii)	It is the angle between the direction vectors $\underset{\sim}{v_1} = \begin{pmatrix}0\\1\\1\end{pmatrix}$ and $\underset{\sim}{v_2} = \begin{pmatrix}1\\3\\2\end{pmatrix}$ $\cos\theta = \frac{\underset{\sim}{v_1}.\underset{\sim}{v_2}}{\lvert\underset{\sim}{v_1}\rvert\lvert\underset{\sim}{v_2}\rvert}$ $\underset{\sim}{v_1}.\underset{\sim}{v_2} = \begin{pmatrix}0\\1\\1\end{pmatrix}.\begin{pmatrix}1\\3\\2\end{pmatrix}$ $= 0\times1 + 1\times3 + 1\times2 = 5$ $\cos\theta = \frac{\underset{\sim}{v_1}.\underset{\sim}{v_2}}{\lvert\underset{\sim}{v_1}\rvert\lvert\underset{\sim}{v_2}\rvert} = \frac{5}{\sqrt{0+1+1}\sqrt{1+9+4}}$ $\theta = 19°6'$	2 mark: Correctly finds the angle between the direction cosines Award 1 mark: Applies to the correct formula, however error in calculation	Angle between two vectors is the angle between the direction vectors, not between the points $\begin{pmatrix}1\\0\\1\end{pmatrix}$ *and* $\begin{pmatrix}2\\0\\0\end{pmatrix}$
c(iii)	Line $\underset{\sim}{r_1}$ passes through the point $A\begin{pmatrix}1\\0\\1\end{pmatrix}$ and let $P\begin{pmatrix}1\\2\\0\end{pmatrix}$ Thus $\vec{AP} = \begin{pmatrix}1-1\\2-0\\0-1\end{pmatrix} = \begin{pmatrix}0\\2\\-1\end{pmatrix}$ $\vec{AP}.\underset{\sim}{v_1} = \begin{pmatrix}0\\2\\-1\end{pmatrix}.\begin{pmatrix}0\\1\\1\end{pmatrix} = 0+2-1 = 1$ $\underset{\sim}{v_1}.\underset{\sim}{v_1} = 2$ Shortest distance = $\left\lvert \vec{AP} - Proj_{v_1}\vec{AP} \right\rvert$ $= \left\lvert \vec{AP} - \frac{\vec{AP}.\underset{\sim}{v_1}}{\underset{\sim}{v_1}.\underset{\sim}{v_1}}\underset{\sim}{v_1} \right\rvert$ $= \left\lvert \begin{pmatrix}0\\2\\-1\end{pmatrix} - \frac{1}{2}\begin{pmatrix}0\\1\\1\end{pmatrix} \right\rvert = \frac{3}{2}\left\lvert \begin{pmatrix}0\\1\\-1\end{pmatrix} \right\rvert = \frac{3}{\sqrt{2}}$	3 marks: correct answer from correct working 2 marks: Applies to the shortest distance formula and minor error 1 mark: Correctly finds $\vec{AP}$ and finds $\vec{AP}.\underset{\sim}{v_1}$ Or Correctly calculates the $Proj_{v_1}\vec{AP}$ 	A diagram would really help. Many students attempted substituting into a formula not realising the meanings of the results they are finding
Question 15			
(a)	As shown in the diagram $f(x) > g(x)$ for $[a, b]$		

	 Area under the curve $y = f(x)$ is greater than area under the curve $y = g(x)$ for $[a, b]$. Hence $$\int_a^b f(x)\,dx > \int_a^b g(x)\,dx$$	2 marks: proves the result providing valid explanation/ working 1 mark: attempts to explain the result, given a diagram or equivalent merit	Well done
(b)(i)	$$I_n = \int_0^1 (1-t^2)^{\frac{n-1}{2}}\,dt,$$ $I_n = \int_0^1 (1-t^2)^{\frac{n-1}{2}}.1\,dt$ $= \left[(1-t^2)^{\frac{n-1}{2}}.t\right]_0^1$ $- \int_0^1 \left(\frac{n-1}{2}\right)(1-t^2)^{\frac{n-3}{2}}(-2t).t\,dt$ **1 mark** $= (n-1)\int_0^1 (1-t^2)^{\frac{n-3}{2}}.t^2\,dt$ $= (n-1)\int_0^1 (1-t^2)^{\frac{n-3}{2}}.[1-(1-t^2)]\,dt$ $= (n-1)\left\{\int_0^1 (1-t^2)^{\frac{n-3}{2}} - (1-t^2)^{\frac{n-1}{2}}\right\}dt$ $I_n = (n-1)\,I_{n-2} - (n-1)\,I_n$ $(1+n-1)I_n = (n-1)\,I_{n-2}$ $\therefore\ n\,I_n = (n-1)\,I_{n-2} \quad n \geq 2$	1 mark: Applies integration by parts 1 mark: manipulates to express the integrand to $(1-t^2)^{\frac{n-\cdots}{2}}$ 2 marks: correctly expresses the integrand in terms of I_n and I_{n-1} and gives the required result 1 mark: Minor error	Well done
(b)(ii)	$n\,I_n = (n-1)\,I_{n-2}$ from (i) Similarly, $(n-1)\,I_{n-1} = (n-2)\,I_{n-3}$ $(n-2)\,I_{n-2} = (n-3)\,I_{n-4}$ **1 mark** . . . Thus, $$J_n = n\,I_n\,I_{n-1}$$ $$= (n-1)I_{n-2} \times \frac{n-2}{n-1}\,I_{n-3}$$ $$= (n-2)I_{n-2}\,I_{n-3}$$ **1 mark** Summarising, $J_n = n\,I_n\,I_{n-1} \quad n \geq 1$ $= (n-2)I_{n-2}\,I_{n-3}$	1 mark: Demonstrates the pattern 1 mark: Demonstrates $J_n = n\,I_n\,I_{n-1}$ $= (n-2)I_{n-2}\,I_{n-3}$ 1 mark: proves $J_n = n\,I_n I_{n-1}$ $= 2I_2\,I_1$ 1 mark:	

	$= (n-4)I_{n-4}\ I_{n-5}$ . . . $= [n-(n-2)]I_2\ I_1$ $= 2\ I_2 I_1$ **1 mark** $I_2 = \int_0^1 (1-t^2)^{\frac{1}{2}}\, dt = \frac{1}{4}\pi \times 1^2 = \frac{\pi}{4}$ $I_1 = \int_0^1 (1-t^2)^0\, dt = 1$ Thus $J_n = 2 \times \frac{\pi}{4} \times 1 = \frac{\pi}{2}$ **1 mark**	Evaluates I_2 and I_1 and proves $J_n = \frac{\pi}{2}$	
(b)(iii)	For $0 \le t \le 1$, $(1-t^2) \ge 0$ **1 mark** Consider the functions $(1-t^2)^{\frac{n-1}{2}}$ and $(1-t^2)^{\frac{n-3}{2}}$ $(1-t^2)^{\frac{n-1}{2}} - (1-t^2)^{\frac{n-3}{2}}$ $(1-t^2)^{\frac{n-3}{2}}[(1-t^2)-1]$ $(1-t^2)^{\frac{n-3}{2}}[(-t^2)]$ For $0 \le t \le 1$, $(1-t^2)^{\frac{n-3}{2}}[(-t^2)] \le 0$ (must prove this) Hence, $(1-t^2)^{\frac{n-1}{2}} < (1-t^2)^{\frac{n-3}{2}}$ **1 mark** Hence, using result in (a), $$\int_0^1 (1-t^2)^{\frac{n-1}{2}}\, dt < \int_0^1 (1-t^2)^{\frac{n-3}{2}}\, dt$$ Thus, $0 < I_n < I_{n-1}$ **1 mark**	1 mark: explains $(1-t^2) \ge 0$ 1 mark: proves $(1-t^2)^{\frac{n-1}{2}} < (1-t^2)^{\frac{n-3}{2}}$ 1 mark: refers to the result in (a) and explains why the inequality holds good.	You must prove the inequalities that you are using. The conditions that are given in the question must be addressed. You should also show reference to the results that you are using from any previous questions
(b)(iv)	From (iii), $I_n < I_{n-1}$ Similarly $I_{n+1} < I_n$ $I_{n+1} < I_n < I_{n-1}$ $I_n > 0$ Thus $I_{n+1}\, I_n < I_n I_n < I_n I_{n-1}$ From (b) (ii), $J_n = n\, I_n\, I_{n-1} = \frac{\pi}{2}$ $I_n\, I_{n-1} = \frac{\pi}{2n}$ $I_{n+1} I_n = \frac{\pi}{2(n+1)}$ Thus, $I_{n+1}\, I_n < (I_n)^2 < I_n I_{n-1}$ $\frac{\pi}{2(n+1)} < (I_n)^2 < \frac{\pi}{2n}$	2 marks: Uses the results in (b)(ii) and (b)(iii) to prove the result. Must show ALL lines of working 1 mark: explains any of the results: $I_n\, I_{n-1} = \frac{\pi}{2n}$ $I_{n+1} I_n = \frac{\pi}{2(n+1)}$ $I_{n+1} < I_n < I_{n-1}$	Generally well done

	$\sqrt{\frac{\pi}{2n+2}} < I_n < \sqrt{\frac{\pi}{2n}}$		

Question 16

(a)(i)	 For $0 \le t \le 20$, $$M\ddot{x} = -\frac{1}{4}M$$ $$\ddot{x} = -\frac{1}{4}$$ For $t > 20$, $$M\ddot{x} = -\frac{1}{4}M - \frac{1}{108}Mv^2$$ $$\ddot{x} = -\frac{1}{108}(27 + v^2)$$	1 mark each : Draws the free body diagrams and writes the force equations
(ii)	$\ddot{x} = -\frac{1}{4} \quad 0 \le t \le 20$ $ie. \ \frac{dv}{dt} = -\frac{1}{4}$ $\int_{60}^{v} dv = -\frac{1}{4}\int_0^t dt$ $v - 60 = -\frac{1}{4}t$ $v = -\frac{1}{4}t + 60 \quad \dots(1)$ $\frac{dx}{dt} = -\frac{1}{4}t + 60$ $\int_0^x dx = \int_0^t -\frac{1}{4}t + 60\, dt$ $x = -\frac{1}{8}t^2 + 60t \quad \dots.(2)$ At $t = 20$, (1)and (2) give, $v = -\frac{1}{4} \times 20 + 60 = 55$ $x = -\frac{1}{8} \times 400 + 60 \times 20 = 1150$ m	2 mark: correctly proves the expression for v and x and substitutes to prove the results 1 mark: Correctly finds the expression for $v \ or \ x$
(iii)	$$\ddot{x} = -\frac{1}{108}(27 + v^2) \quad t \ge 20$$ $$ie. \ v\frac{dv}{dx} = -\frac{1}{108}(27 + v^2)$$ $$\frac{vdv}{27 + v^2} = -\frac{1}{108}dx$$ $\int_{55}^{v} \frac{vdv}{27 + v^2} = -\frac{1}{108}\int_{1150}^{x} dx$ $\left[\frac{1}{2}\log(27 + v^2)\right]_{55}^{v}$ $= -\frac{1}{108}[x]_{1150}^{x}$	2 mark: proves the result 1 mark: Writes the expression $$v\frac{dv}{dx} = -\frac{1}{108}(27 + v^2)$$ And attempts to prove the result

	$\frac{1}{2}\log\left[\frac{27+v^2}{27+55^2}\right] = -\frac{1}{108}(x-1150)$ $\therefore\ x - 1150 = 54\log\left[\frac{27+55^2}{27+v^2}\right]$ $x = 1150 + 54\,\{\ln(27+(55)^2) - \ln(27+v^2)\}$	
(iv)	The jet comes to rest when $v = 0$. Ie. $x = 1150 + 54\,\{\ln(27+(55)^2) - \ln(27+v^2)\}$ $x = 1150 + 54\,\{\ln(27+(55)^2) - \ln(27)\}$ $= 1405.296638\ldots$ $= \ 1405\ m$ The jet comes to rest 1405 m from the touchdown point.	1 mark: Substitutes $v = 0$ and gets the x value.

(bi)	$1-t+t^2-t^3+\cdots+(-1)^{n-1}t^{n-1}$ GP with $a=1, r=-t$ and n terms $$S_n = a\left(\frac{1-(-t)^n}{1-(-t)}\right)$$ $$= \frac{1}{1+t}-(-1)^n\frac{t^n}{1+t}$$ Thus, $$1-t+t^2-t^3+\cdots+(-1)^{n-1}t^{n-1} = \frac{1}{1+t}-(-1)^n\frac{t^n}{1+t}$$ Integrating both sides, 0 to x, $$\int_0^x 1-t+t^2-t^3+\cdots+(-1)^{n-1}t^{n-1}dt = \int_0^x \frac{1}{1+t}-(-1)^n\frac{t^n}{1+t}dt$$ $$\left[t-\frac{t^2}{2}+\frac{t^3}{3}+\cdots+(-1)^{n-1}\frac{t^n}{n}\right]_0^x = [\ln(1+t)]_0^x - \int_0^x (-1)^n\frac{t^n}{1+t}dt$$	1 mark: finds the sum of GP correctly. 1 mark: Integrates both sides of the equality 0 to x, proves the result
	For $0<t<x$, $1<1+t$ $Hence,\quad \frac{1}{1+t}<1$ **1 mark** Also, $\frac{t^n}{1+t}<t^n$ Thus $\int_0^x \frac{t^n}{1+t}dt < \int_0^x t^n dt$ **1 mark**	1 mark: proves the inequality $\frac{1}{1+t}<1$ 1 mark: proves the result
(iii)	$\int_0^x \frac{t^n}{1+t}dt < \int_0^x t^n dt = \left[\frac{t^{n+1}}{n+1}\right]_0^x = \frac{x^{n+1}}{n+1}$ Thus for $0<x<1$, Hence $$0<\frac{x^{n+1}}{n+1}<\frac{1^{n+1}}{n+1}=\frac{1}{n+1}$$ $$\int_0^x \frac{t^n}{1+t}dt < \frac{1}{n+1}$$	
(iv)	From (i), $$x-\frac{x^2}{2}+\frac{x^3}{3}+\cdots+(-1)^n\frac{x^n}{n} = \log_e(1+x)-(-1)^n\int_0^x\frac{t^n}{1+t}dt$$ From (ii), when $n\to\infty$, $$\int_0^x \frac{t^n}{1+t}dt < \frac{1}{n+1}\to 0$$ Thus as $n\to\infty$, $$x-\frac{x^2}{2}+\frac{x^3}{3}+\cdots+(-1)^n\frac{x^n}{n}+\cdots = \log_e(1+x)$$	2 marks: correct proof with references to previous results and explanation 1 mark: minor error